Int. J. Solids Structures, 1970, Vol. 6, pp. 617 to 639. Pergamon Press. Printed in Great Britain

INTERACTION BETWEEN STRUCTURES AND BILINEAR FLUIDS†

H. H. BLEICH and I. S. SANDLER

Department of Civil Engineering and Engineering Mechanics, Columbia University, New York

Abstract—As a mathematical model of fluids in which cavitation may occur, wave propagation in bilinear fluids is investigated. The principal difficulty is the determination of the *a priori* unknown and moving boundaries between the two stages of the fluid.

A characteristic approach is used to study one-dimensional problems. Even in such relatively simple situations a considerable variety of cases may occur and a lengthy analysis is required. The paper discusses also the non-trivial questions of uniqueness and existence of solutions for the bilinear model.

The procedure developed is applied to a typical example.

NOTATION

c_1	sound speed in the high-density material
c_2	sound speed in the low-density material
g	acceleration of gravity
ε	small positive number
L	length used to define non-dimensional variables
m	non-dimensional variable related to the momentum
P, p	actual and non-dimensional pressure
P_0	pressure at which the changeover between high- and low-density material occurs
T, t	actual and non-dimensional time
V	velocity of an interface (non-dimensional)
v	component of particle velocity in the X-direction
X	spatial coordinate
$x = X/L$	non-dimensional spatial variable
$\Delta t, \Delta x$	time and space increments
β	ratio of sound speeds c_2/c_1
γ	density
γ_0	density at which the changeover between high- and low-density material occurs
ρ	non-dimensional variation in the density

1. INTRODUCTION

THIS investigation is motivated by interest in problems concerning the effect of shock waves on structures submerged in a fluid or floating on a liquid half-space. While there is an extensive literature [1] on such problems, the cases treated ignore cavitation, and are therefore limited to values of applied shock intensities below levels for which cavitation will occur. As a first step towards an understanding of the phenomena, one may conclude intuitively that the principal influence of cavitation on a structural target will be caused by the fact that in a cavitated region the pressure will be roughly uniform, equal to a

† This investigation was sponsored by the Office of Naval Research under Contract No. 266(86) with Columbia University.

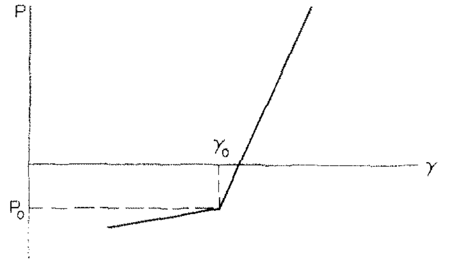

FIG. 1(a).

nominal cavitation pressure P_0. This leads to the pressure–density relation represented by Fig. 1(b). Because of lack of knowledge concerning the mathematical consequences of assuming this degenerate relation in the analysis, the case allowing for a slight linear variation of the cavitation pressure with density, Fig. 1(a), will be studied.

The assumption that the cavity pressure is uniform is widely used in problems of steady flow with cavitation. For transient situations, it has also been discussed in [2], and is utilized in [3] and, in a modified manner, in [4]. The latter two references appear to be the most advanced investigations available concerning dynamic problems with cavitation. Each considers one special example of the general problem treated here. Both references treat the formation and expansion of the cavitated region in a manner suitable for these special cases, giving for these early stages results equivalent to the ones obtained from the approach in this paper. However, in the subsequent stage when the cavitated region contracts, the analysis becomes much more complex. This situation is treated in the references only by use of crude approximations.

The references quoted do not consider or discuss the great variety of possibilities which exist in general in the behavior of the boundaries of cavitated regions. Moreover, the important questions of existence and uniqueness of the solutions of the basic equations are not mentioned or explored in either reference.

The present paper is thus concerned with an idealized problem, the propagation of pressure waves and weak shocks in an inviscid liquid which exhibits a bilinear pressure–density relation of the type shown in Fig. 1(a). The relation between the pressure P and the density γ is "hard" for values of the density above a critical value γ_0 and "soft" for values below γ_0. This idealized problem was treated in general in [5]. Only certain of the more important details and results appearing in [5] will be presented in this paper.

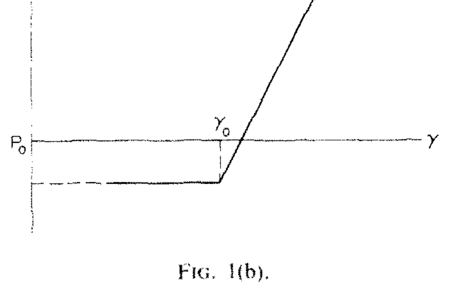

FIG. 1(b).

While the results are obviously applicable to the numerical solution of problems by characteristic methods, the emphasis of the treatment is on the recognition of the physical situations which may arise, and on the matters of uniqueness and existence of solutions. The reason for such an emphasis is the possibility of devising finite difference schemes for bilinear models in which not only shocks but also the boundaries between hard and soft regions are smoothed out. Such schemes furnish numerical results which might be meaningless if the mathematical formulation prior to the introduction of finite differences permits several solutions or none. Because of the complexity of the analysis, only the one-dimensional case is considered.

In similar problems of wave propagation when the pressure–density relation is non-linear but smooth, the theory of hyperbolic differential equations gives the proofs of existence and uniqueness, at least in the small, provided physical considerations are used to exclude certain excess solutions [6, 7]. In the bilinear case this theory is not applicable, because of the occurrence of *a priori* unknown moving interfaces between hard and soft regions.

The investigation utilizes characteristic methods, the crux being consideration of the situations at the interfaces. The field at these interfaces may be continuous, or may have discontinuities requiring separate treatment. Additional special cases are caused by the fact that regions in which $\gamma \gtrless \gamma_0$ can form or disappear, i.e. interfaces, considered as functions of time, have starting and end points. Such points correspond to singularities in the solution, which may be severe when special interfaces start in the same location.

The degenerate case of a pressure–density relation according to Fig. 1(b) is considered as a limit of the general bilinear case. It is found in [5] that the degenerate case remains physically meaningful only with restrictions.

2. BASIC EQUATIONS OF THE MOTION OF THE FLUID

The behavior of the medium to be studied is defined by two different but linear pressure–density relations. These relations, and the regions in which they apply, will be referred to as high-density, $\gamma > \gamma_0$, and low-density, $\gamma < \gamma_0$, respectively, where γ_0 is the density which defines the kink in Figs. 1(a) and 1(b). The two laws are

$$P = \begin{cases} P_0 + c_1^2(\gamma - \gamma_0) & \text{for } \gamma \geq \gamma_0 \\ P_0 + c_2^2(\gamma - \gamma_0) & \text{for } \gamma \leq \gamma_0 \end{cases} \tag{2.1}$$

where the constants $c_1 > c_2$ may be interpreted as local sound speeds if $c_2 \neq 0$.

The continuity and momentum equations for one-dimensional flow in the X-direction for an inviscid fluid are

$$\begin{aligned} \frac{\mathrm{D}\gamma}{\mathrm{D}T} + \gamma\frac{\partial v}{\partial X} &= 0 \\ -\frac{\partial P}{\partial X} + \gamma f &= \gamma\frac{\mathrm{D}v}{\mathrm{D}T} \end{aligned} \tag{2.2}$$

where v and f are velocity and body force components in the X-direction and $\mathrm{D}/\mathrm{D}T$ denotes material or total time differentiation. The analysis will be restricted to situations in which the density changes are small†

$$\left|\frac{\gamma-\gamma_0}{\gamma_0}\right| \ll 1 \tag{2.3}$$

and where the velocities are small, $|v| \ll c_2 < c_1$. By introducing the non-dimensional variables

$$x = \frac{X}{L} \qquad t = \frac{c_1 T}{L} \qquad p = \frac{P-P_0}{\gamma_0 c_1^2}$$
$$\rho = \frac{\gamma-\gamma_0}{\gamma_0} \qquad m = \frac{\gamma v-\gamma_0 f T}{\gamma_0 c_1} \tag{2.4}$$

where L is a reference length, one obtains

$$\dot{\rho}+m' = 0 \tag{2.5}$$

$$\dot{m}+p' = 0 \tag{2.6}$$

$$p = \begin{cases} \rho & \text{for } \rho \geq 0 \\ \beta^2\rho & \text{for } \rho \leq 0 \end{cases} \tag{2.7}$$

where the primes and dots denote partial differentiation with respect to x and t, respectively, and where $\beta = c_2/c_1$. The inequality $0 \leq c_2 < c_1$, leads to the limitation $0 \leq \beta < 1$. It is also noted that equation (2.3) requires $|\rho| \ll 1$.

The construction of the solution, uniqueness, and existence for the bilinear material will be examined by characteristic methods, the appropriate relations being

$$\left.\begin{aligned} m-\rho &= \text{const. on } x+t = \text{const.} \\ m+\rho &= \text{const. on } x-t = \text{const.} \end{aligned}\right\} \text{ for } \rho > 0 \tag{2.8}$$

$$\left.\begin{aligned} m-\beta\rho &= \text{const. on } x+\beta t = \text{const.} \\ m+\beta\rho &= \text{const. on } x-\beta t = \text{const.} \end{aligned}\right\} \text{ for } \rho < 0 \tag{2.9}$$

where $\beta > 0$. If the material in the vicinity of any point 0 is entirely of high- or entirely of low-density, the established proofs for the uniform material apply and there is no problem. If both types of material occur in such a vicinity, a more general approach for the simultaneous determination of the interfaces and of the solution must be developed. The construction of the solution is treated in steps, i.e. in initial value situations the solution for $t = t_0+\Delta t$ is obtained from the solution at $t = t_0$. There are multitude of separate cases to be studied depending on the configuration of the interfaces.

The analysis of initial value situations in [5] uses an inverse procedure, in steps. First, for each possible configuration, i.e. location of interfaces, a set of necessary conditions on the initial prescriptions is obtained. The second step is the proof that the necessary conditions are sufficient to guarantee existence of one and only one solution of the configuration studied. At a third, but later stage, the matters of uniqueness and existence require further examination to show that for any set of initial values one and only one solution exists.

† Otherwise the linear relations (2.1) are physically unrealistic.

3. REGIONS WITH CONTINUOUS INITIAL PRESCRIPTIONS CONTAINING A SINGLE INTERFACE

If a small but finite one-dimensional region contains both high- and low-density locations, the line $0L$ separating these locations in the x–t plane, Fig. 2(a), will be called an "interface". As a special case such a line-interface may degenerate into a "neutral region" where $\rho \equiv 0$, Fig. 2(b).

All possible inclinations for a single interface or for several intersecting interfaces in the vicinity of any point 0 in the x–t plane must be considered. The present section deals with the continuation of the solution for given continuous initial values on the premise that the continuation contains just one interface. Even if the initial prescription is continuous, the solution may subsequently develop a discontinuity at the interface,† so that both continuous and discontinuous solutions have to be treated.

The mathematically permissible solutions are ultimately accepted only if they also pass a physical test, namely the requirement that mechanical energy be either conserved or dissipated, but not created. This energy requirement eliminates interfaces which are rarefaction shocks.

(a) *Continuations having one interface without discontinuity*

Let the curve $0L$ in Fig. 2(a) be a small but finite section of an interface defined by two values of the time, t_0 and $t_0+\Delta t$. Conditions on continuous initial prescriptions at $t = t_0$ near point 0 will be given which permit continuation of the solution having just one interface without discontinuity. Along such an interface the density ρ must vanish, $\rho \equiv 0$, while

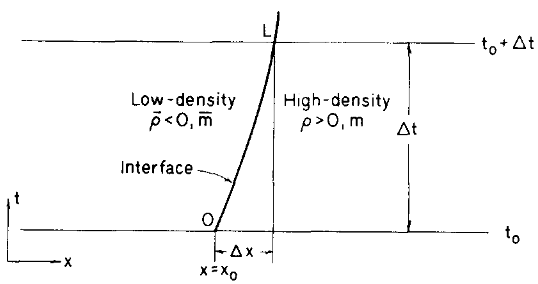

FIG. 2(a).

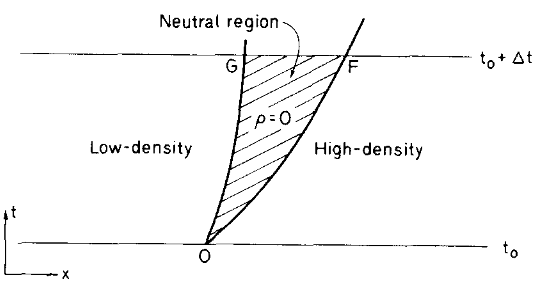

FIG. 2(b).

† In this respect the material exhibits a behavior similar to that of a work-hardening material with a smooth $p-\rho$ relation, where the possibility of shock formation is well known.

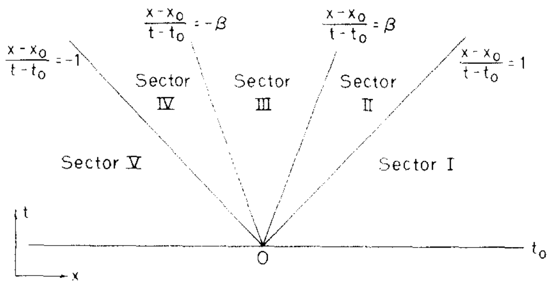

FIG. 3. Definition of Sectors I–V.

the quanity m is continuous,† $m = \bar{m}$. Further, in the immediate vicinity of $0L$ the inequalities $\rho > 0$, $\bar{\rho} < 0$ must hold for $t_0 < t \leq t_0+\Delta t$, otherwise $0L$ would not be an interface. (The conditions $\rho > 0$, $\bar{\rho} < 0$ may not apply for $t = t_0$, where the prescriptions $\rho = 0$ and/or $\bar{\rho} = 0$ may still lead to the formation of an interface for $t > t_0$.)

The details of the analysis depend on the location and slope of the interface with respect to the characteristic directions, $dx/dt = \pm\beta$ and ± 1, Fig. 3. Six cases, listed below, are to be considered. In Cases 1, 3 and 5 the interface lies entirely within the respective sector as defined in Fig. 3. For these cases an interface is still considered to lie within a sector if it is tangent at point 0 to one of the boundaries $dx/dt = \pm 1$, $\pm\beta$ or $\pm\infty$ but does not coincide with that boundary line. In Cases 2 and 4 the interface may either be within its respective sector, or on one of the straight boundary lines of its sector. Case 6, finally, concerns neutral regions of unspecified location having two boundaries intersecting at point 0 such as $0F$ and $0G$ in Fig. 2(b). The cases to be treated are thus:

Case (1) Interface in Sector I.
Case (2) Interface in Sector II or on one of its boundary lines, $(x-x_0)/(t-t_0) = +\beta$ or $+1$.
Case (3) Interface in Sector III.
Case (4) Interface in Sector IV or on one of its boundary lines, $(x-x_0)/(t-t_0) = -\beta$ or -1.
Case (5) Interface in Sector V.
Case (6) Neutral region.

For reasons of brevity only Case 1 on the above list will be extensively presented. The procedure in the other five cases may be found in [5].

Case 1. Interface without discontinuity situated in Sector I, Fig. 4(a). If the curve $0A$ in Fig. 4(a) is an interface without discontinuity, the solution in the high-density material (to the right of $0A$ and denoted by H.D.M.) is entirely defined by the initial values to the right of point 0. These values must be such that the density in the region $A02$ satisfies the premise $\rho \geq 0$, where the equal sign, $\rho \equiv 0$, applies on the curve $0A$, while the inequality $\rho > 0$ applies immediately (i.e. within a sufficiently narrow strip) to the right of the curve $0A$. In addition it is necessary that $\bar{\rho} < 0$ immediately to the left of $0A$, while $\bar{\rho} \leq 0$ in the region $A03$, so that no second interface occurs.

The analysis is restricted to initially prescribed densities and momenta which are continuous and behave monotonically within some possibly small but finite region on each side of point 0. The characteristic relations (2.8, 9) then imply that the values of ρ and m

† The symbols ρ, m and $\bar{\rho}$, $\bar{m}$ refer to the density and momentum in the high- and low-density regions, respectively.

throughout each of the regions $A02$, $A0P$, $P0Q$ and $Q03$ must also be continuous and monotonic. Due to this monotonic behavior of the density ρ in the region $A02$, the requirement $\rho > 0$ immediately to the right of $0A$ may be restated as $\rho(x) > 0$ for $x_0 < x \leq x_0 + \varepsilon_1$, where ε_1 is a possibly small but finite positive number. The additional requirement $\rho_A = 0$, in conjunction with the characteristic relations (2.8) along lines $A1$ and $A2$, gives

$$\begin{aligned} m_A &= m_1 + \rho_1 \\ m_A &= m_2 - \rho_2. \end{aligned} \tag{3.1}$$

Eliminating m_A from equations (3.1) and noting $\rho_1 > 0$, one may write

$$m_2 - \rho_2 > m_1 - \rho_1. \tag{3.2}$$

In view of the monotonic behavior of the function $(m - \rho)$ along the line 02, equation (3.2) and the requirement $\rho_0 = 0$ imply

$$m_2 - \rho_2 > m_0. \tag{3.3}$$

Equation (3.3), in conjunction with $\rho_2 > 0$, permits the conclusion that the initially prescribed functions $m(x)$ and $\rho(x)$ must satisfy the conditions

$$\left.\begin{aligned} m(x) - m_0 - \rho(x) > 0 \\ \rho(x) > 0 \end{aligned}\right\} x_0 < x \leq x_0 + \varepsilon \tag{3.4}$$

throughout a possibly small but finite domain defined by the quantity $\varepsilon \leq \varepsilon_1$. Equations (3.4) are necessary conditions for the present case.

Additional necessary conditions follow from consideration of the low density region. The condition $\rho_A = 0$ and equations (3.1), in view of the continuity of the initial prescription, indicate that the density and momentum are continuous along the curve $0A$. The theory of linear hyperbolic partial differential equations then permits the conclusion that a continuation of the solution based on the differential equations for the low-density material in the region $A03$ is necessarily continuous. In particular, no discontinuities may occur along the characteristics $0P$ and $0Q$ because the prescription at point 0 is continuous along the line $A03$. Therefore the requirements that $\bar{\rho} < 0$ immediately to the left of line $0A$ and that $\bar{\rho} \leq 0$ throughout the region $A03$, may be restated as $\bar{\rho}_P < 0$, $\bar{\rho}_Q \leq 0$ and $\bar{\rho}_3 \leq 0$. The last inequality gives simply the condition $\bar{\rho}(x) \leq 0$ for $x_0 - \bar{\varepsilon} \leq x < x_0$, where $\bar{\varepsilon} > 0$.

The condition $\bar{\rho}_P < 0$ is considered next. Eliminating $\bar{m}_P$ from the characteristic relations along $0P$ and PA leads to

$$\bar{\rho}_P = -\frac{m_2 - m_0 - \rho_2}{2\beta} \tag{3.5}$$

which, in view of equation (3.3), indicates that the requirement $\bar{\rho}_P < 0$ is automatically satisfied.

To treat the requirement $\bar{\rho}_Q \leq 0$, the momentum $\bar{m}_Q$ is eliminated from the characteristic relation along $0Q$ and $Q3$ to obtain

$$2\beta\bar{\rho}_Q = \beta\bar{\rho}_3 + \bar{m}_3 - m_0. \tag{3.6}$$

Introduction of the requirement $\bar{\rho}_Q \leq 0$ gives

$$\bar{m}_3 - m_0 + \beta\bar{\rho}_3 \leq 0 \tag{3.7}$$

which, in conjunction with $\bar{\rho}_3 \leq 0$, permits the conclusion that the initially prescribed functions $\bar{m}$ and $\bar{\rho}$ must satisfy the conditions

$$\left.\begin{aligned} \bar{m}(x) - m_0 + \beta\bar{\rho}(x) \leq 0 \\ \bar{\rho}(x) \leq 0 \end{aligned}\right\} x_0 - \bar{\varepsilon} \leq x < x_0. \tag{3.8}$$

Equations (3.8) were derived from the requirements $\bar{\rho}_Q \leq 0$ and $\bar{\rho}_3 \leq 0$ which assure that no second interface occurs. It can easily be shown from the appropriate characteristic relations that the converse is true, i.e. that equations (3.8) insure that the conditions $\bar{\rho}_Q \leq 0$ and $\bar{\rho}_3 \leq 0$ are satisfied. As it has already been shown that the condition $\bar{\rho}_P < 0$ is a consequence of equation (3.4), the relations (3.4, 8) together form not only a set of necessary conditions but a set of sufficient conditions, provided that an interface $0A$ in Sector 1 exists.

To demonstrate that equations (3.4, 8) guarantee the existence of the interface in Sector I for a possibly small but finite interval of time, let point 2 be an arbitrary point to the right of, but sufficiently close to point 0, Fig. 4(a). Equations (3.4) imply then the inequalities

$$0 < m_2 - m_0 - \rho_2 < m_2 - m_0 + \rho_2. \tag{3.9}$$

Since the initial values have been assumed to be continuous and monotonic, equation (3.9) implies that the function $[m(x) - m_0 + \rho(x)]$ vanishes at $x = x_0$ and is positive, continuous and monotonic in a finite domain $x_0 < x_2 \leq x_0 + \varepsilon$. Noting $\rho_2 > 0$, there exists

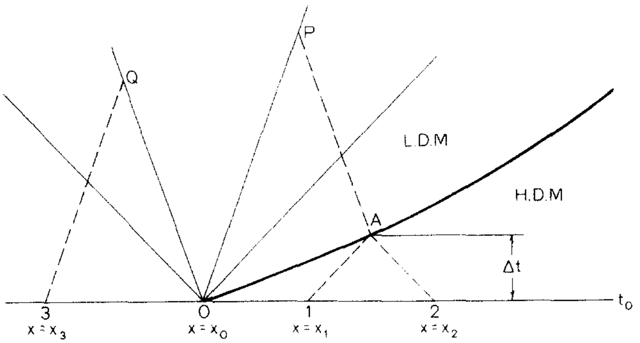

FIG. 4(a). Case 1.

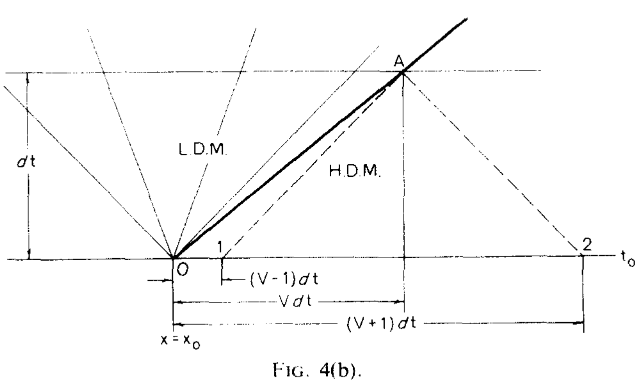

FIG. 4(b).

thus a unique point 1 on the initial line between points 0 and 2 (not coinciding with either) such that

$$m_1 - m_0 + \rho_1 = m_2 - m_0 - \rho_2. \tag{3.10}$$

Equations (3.1) can thus be satisfied and the point A may be determined uniquely from points 1 and 2. By varying the initial point 2, the above considerations indicate that the locus of points A defines a curve entirely in Sector I which is unique for any set of initial values satisfying equations (3.4, 8). However, this curve can be accepted as an interface only if its derivative, the velocity, satisfies $dx/dt > 1$ at all points between 0 and A, because the characteristic construction would otherwise not be valid. This condition may not be satisfied for the entire range of points 2 for which equation (3.4) hold, but will be satisfied for a sufficiently small range, because it was found that point 1 lies to the right of point 0.

Due to the existence of the interface 0A on which $\rho = 0$, the conventional uniqueness and existence theorems for linear partial differential equations apply, and the solution in the high-density region exists and can be uniquely determined from the prescription on $x \geq x_0$. This furnishes values of m on 0A which together with the prescription on $x < x_0$ similarly define the solution in the low-density region. Conditions (3.4, 8) are thus seen to be necessary and sufficient for the continuation of the solution of the type considered.

The above considerations do not yet answer the question as to whether alternate solutions with interfaces in other sectors might exist for initial prescriptions satisfying equations (3.4, 8). This matter will be considered later.

Conditions (3.4, 8) are not convenient for use in numerical computations in which the initial functions $m(x)$, $\rho(x)$, $\bar{m}(x)$ and $\bar{\rho}(x)$ are known only at isolated points. In such cases each of the initial functions is assumed expandable in a power series on the appropriate side of point 0. Truncation of the expansions (retaining only first derivatives) and substitution into equations (3.4, 8) leads to an alternative set of sufficient conditions

$$\begin{gathered} m' > \rho' > 0 \\ \bar{m}' > \beta\rho' > 0 \\ \bar{\rho}' > 0 \end{gathered} \tag{3.11}$$

where m', ρ', $\bar{m}'$ and $\bar{\rho}'$ are the one-sided derivatives at point 0 of $m(x)$, $\rho(x)$, $\bar{m}(x)$, $\bar{\rho}(x)$, respectively. The inequalities (3.11) are somewhat more restrictive than equations (3.4, 8) and, while sufficient, are not necessary conditions.

In numerical computations it may also be desirable to have an expression for the velocity of the interface. To find the velocity $V = V(t_0)$ at point 0 one may use an asymptotic approach in which the conditions in the neighborhood of point 0, Fig. 4(a), are considered only in the limit as $\Delta t — 0$. For this purpose reference is made to Fig. 4(b) in which the time difference Δt is replaced by the time differential dt. Noting the locations of points 1 and 2 in Fig. 4(b), substitution of the truncated series expansions for $m(x)$ and $\rho(x)$ into equation (3.10) gives

$$V = V(t_0) = \frac{m'}{\rho'}. \tag{3.12}$$

Cases 2 through 6. It is demonstrated in [5] that Cases 2 and 4 lead to contradictions, so that of the Cases 2 to 6 only Cases 3, 5 and 6 can occur. The results of the analysis presented in [5] for these cases (as well as some other cases to be discussed later) are summarized in

Table 1(a). Simplified (sufficient) conditions in terms of first derivatives $[\rho'(x_0) > 0, \bar{\rho}'(x_0) > 0]$

	$\frac{m'}{\rho'} < 1$	$\frac{m'}{\rho'} > 1$	
$\frac{\bar{m}'}{\bar{\rho}'} > -1$	Case 3 $\frac{\bar{m}'}{\bar{\rho}'} > -\beta + \frac{1-\beta}{2\beta}\frac{(\rho'-m')}{\bar{\rho}'}$	Case 1	$\frac{\bar{m}'}{\bar{\rho}'} > -\beta$
	Case 7 $\frac{\bar{m}'}{\bar{\rho}'} < -\beta + \frac{1-\beta}{2\beta}\frac{(\rho'-m')}{\bar{\rho}'}$	Case 9 if $\frac{\bar{\rho}'}{\rho'} > z\left(\frac{m'}{\rho'}, \frac{\bar{m}'}{\bar{\rho}'}\right)$; Case 11 if $\frac{\bar{\rho}'}{\rho'} < z\left(\frac{m'}{\rho'}, \frac{\bar{m}'}{\bar{\rho}'}\right)$	$-\beta > \frac{\bar{m}'}{\bar{\rho}'} > -1$
$\frac{\bar{m}'}{\bar{\rho}'} < -1$	Case 5	Case 10 if $\frac{\bar{\rho}'}{\rho'} > z_1\left(\frac{m'}{\rho'}, \frac{\bar{m}'}{\bar{\rho}'}\right)$; Case 12 if $\frac{\bar{\rho}'}{\rho'} < z_1\left(\frac{m'}{\rho'}, \frac{\bar{m}'}{\bar{\rho}'}\right)$	$\frac{\bar{m}'}{\bar{\rho}'} < -1$
The functions $z\left(\frac{m'}{\rho'}, \frac{\bar{m}'}{\bar{\rho}'}\right)$ and $z_1\left(\frac{m'}{\rho'}, \frac{\bar{m}'}{\bar{\rho}'}\right)$ are defined in [5].			

Table 1(b). Necessary and sufficient conditions provided $\rho(x) > 0$, $\bar{\rho}(\bar{x}) < 0$

	$q(x) < 0$	$q(x) = 0$	$q(x) > 0$	
$\bar{q}(\bar{x}) < \bar{\bar{q}}(\bar{\bar{x}})$	Case 3 $q(x) > \bar{q}(\bar{x})$; $\bar{q}(\bar{x}) = q(x), (V = -\beta)$; Case 7 $q(x) \leq \bar{q}(\bar{x})$, $\bar{q}(\bar{x}) \leq \bar{\bar{q}}(\bar{\bar{x}})$	Case 6	Case 1	$\bar{q}(\bar{x}) < 0$
		$\bar{q}(\bar{x}) = q(x) = 0$		$\bar{q}(\bar{x}) = 0$
			See Table Ia	$\bar{q}(\bar{x}) > 0$
$\bar{q}(\bar{x}) = \bar{\bar{q}}(\bar{\bar{x}})$	$\bar{q}(\bar{x}) = \bar{\bar{q}}(\bar{\bar{x}})$; $(V = -1)$			
$\bar{q}(\bar{x}) > \bar{\bar{q}}(\bar{\bar{x}})$	Case 5			

Tables 1(a) and 1(b). The former gives results in terms of first derivatives, and the latter gives the general results for initial prescriptions where $\rho(x) > 0$ and $\bar{\rho}(x) < 0$. The ranges of applicability of the various solutions in Table 1(b) are defined by inequalities or equalities on the functions

$$
\begin{aligned}
q(x) &= m(x) - m_0 - \rho(x), && x_0 < x \leq x_0 + \varepsilon \\
\bar{q}(\bar{x}) &= \bar{m}(\bar{x}) - m_0 + \beta\bar{\rho}(\bar{x}) && \\
\bar{\bar{q}}(\bar{\bar{x}}) &= \bar{m}(\bar{\bar{x}}) - m_0 - \beta\bar{\rho}(\bar{\bar{x}}) && \bar{x} < \bar{\bar{x}} < x_0.
\end{aligned}
\tag{3.13}
$$

These functions are combinations of the initially prescribed densities and momenta at three related points $x > x_0$, $\bar{x} < x_0$ and $\bar{\bar{x}} < x_0$ in the (finite) neighborhood of the point 0 in which the solution is to be obtained. The locations $\bar{x}$ and $\bar{\bar{x}}$ are functions of x_0 and of the generic point x,

$$\begin{aligned} \bar{x} &= x_0 - \frac{2\beta}{1-\beta}(x - x_0) \\ \bar{\bar{x}} &= x_0 - \frac{1-\beta}{1+\beta}(x_0 - \bar{x}). \end{aligned} \tag{3.14}$$

(b) *Interfaces with discontinuity*

Up to this point a variety of cases have been considered for which the continuation of the solution contains either an isolated interface without discontinuity or a neutral region. It is now necessary to determine whether the continuation of the solution may contain an interface with a discontinuity even when the initial prescription is continuous. The situation is much more complex than in cases without discontinuity.

Prior to the determination of specific initial prescriptions for which discontinuities occur, the general character of interfaces with discontinuities will be considered. At such an interface the two separate solutions of the differential equations in the adjoining high- and low-density regions must satisfy the physical conservation laws. Figure 5 shows the

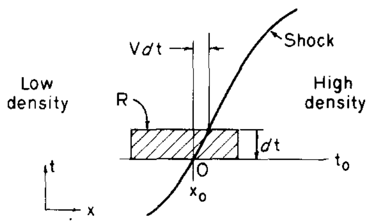

FIG. 5.

interface in the x–t plane, its velocity at point 0 being V. The relevant physical laws allow for two types of discontinuities—contact discontinuities and shocks. An interface of the former type need not be considered here because it is incompatible with the pressure–density relation (2.1) for $\beta \neq 0$ (a contact discontinuity requires that the pressure be continuous while the density is not). For a shock, the relations expressing the conservation of mass and momentum, when used in conjunction with the assumption that the particle velocities are small compared to V, give

$$V = \frac{m - \bar{m}}{\rho - \bar{\rho}} = \frac{p - \bar{p}}{m - \bar{m}}. \tag{3.15}$$

Using equation (2.7) this becomes

$$V = \pm \left| \frac{\rho - \beta^2 \bar{\rho}}{\rho - \bar{\rho}} \right| \tag{3.16}$$

which limits V to the range

$$\beta \leq |V| \leq 1. \tag{3.17}$$

Before studying solutions with discontinuities at the interfaces it is appropriate to check whether their occurrence does not violate energy considerations.† It was shown in [5] that positive velocities V which represent rarefaction shocks are not permissible, except when $V = +\beta$ or $+1$. Only these two positive velocities, resulting in rarefaction shocks, do not violate energy considerations. However, these two velocities cannot occur for continuous initial prescriptions, as shown in [5].

Case 7. Interface with discontinuity situated in sector IV or on one of its boundary lines, Fig. 6. To determine the initial prescriptions for which such an interface occurs, the values

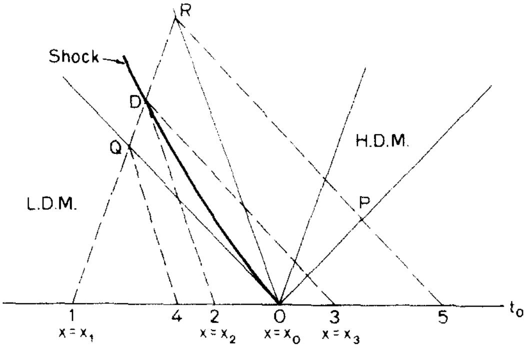

FIG. 6. Case 7.

ρ_D, m_D and $\bar{\rho}_D$, $\bar{m}_D$ applying on opposite sides of the interface at point D, Fig. 6, must be distinguished. One may then write the characteristic relations

$$\begin{aligned}\bar{m}_D+\beta\bar{\rho}_D &= \bar{m}_1+\beta\bar{\rho}_1\\ \bar{m}_D-\beta\bar{\rho}_D &= \bar{m}_2-\beta\bar{\rho}_2\\ m_D-\rho_D &= m_3-\rho_3\end{aligned} \tag{3.18}$$

and the shock relations, (3.15, 16),

$$\begin{aligned}V(t) &= \frac{m_D-\bar{m}_D}{\rho_D-\bar{\rho}_D}\\ V(t) &= -\left|\sqrt{\left(\frac{\rho_D-\beta^2\bar{\rho}_D}{\rho_D-\bar{\rho}_D}\right)}\right|.\end{aligned} \tag{3.19}$$

Several requirements will be formulated which must be satisfied at various points, Fig. 6. Excluding temporarily the possibility that the interface coincides with one of the straight boundary lines $0Q$ or $0R$, the requirements to be considered are $\bar{\rho}_1 \leq 0$, $\bar{\rho}_D < 0$, $\rho_D > 0$, $\rho_P \geq 0$ and $\rho_5 \geq 0$. (The possibilities $\bar{\rho}_D = 0$, $\rho_D = 0$ are not allowed because the second of equation (3.19) indicates that they correspond to the temporarily excluded cases $V(t) = -\beta$, -1 when the interface coincides with $0Q$ or $0R$, respectively.) In addition to these five relations, the velocity of the interface at point D is thus subject to the inequality

† Considerations based on entropy changes, usually invoked in the study of shocks, are not applicable because the model for the material in this study ignores changes in temperature. As a result, the model used permits the occurrence of (weak) rarefaction shocks within a high-density or within a low-density region.

$-1 < V(t) < -\beta$. Introduction of the bound $V(t) < -\beta$ into the first of the equation (3.19) requires

$$m_D - \bar{m}_D < -\beta(\rho_D - \bar{\rho}_D). \tag{3.20}$$

Due to the monotonic behavior of the solution in each of the sectors $D01$, $D0P$ and $P05$, the above requirements assure that the solution complies with the premise of a single interface. These requirements will be used to obtain a set of necessary conditions.

The inequality (3.20) may be rearranged,

$$m_D + \beta\rho_D < \bar{m}_D + \beta\bar{\rho}_D \tag{3.21}$$

which gives, after use of $\rho_D > 0$ and of the first and third of equation (3.18),

$$\bar{m}_1 + \beta\bar{\rho}_1 > m_D - \rho_D = m_3 - \rho_3. \tag{3.22}$$

The last inequality may be strengthened as follows. Due to the monotonic behavior of the solution on the line $1QD$, the requirements $\bar{\rho}_1 \leq 0$. $\bar{\rho}_D < 0$ permit the conclusion $\bar{\rho}_Q < 0$. This inequality and the requirement $\rho_P \geq 0$, after use of the characteristic relations along $Q1$, $Q4$ and $0P$, $P5$ give respectively

$$\begin{aligned} \bar{m}_1 + \beta\bar{\rho}_1 &< \bar{m}_4 - \beta\bar{\rho}_4 \\ m_5 - \rho_5 &\leq m_0. \end{aligned} \tag{3.23}$$

The second of equations (3.23) and monotonic behavior imply

$$m_5 - \rho_5 \leq m_3 - \rho_3 \leq m_0 \tag{3.24}$$

which, when substituted into equation (3.22) yields finally

$$\bar{m}_1 + \beta\bar{\rho}_1 > m_5 - \rho_5. \tag{3.25}$$

Equations (3.23, 25) in conjunction with $\bar{\rho}_1 \leq 0$ and $\rho_5 \geq 0$ permit thus the conclusion that the following conditions are necessary:

$$\begin{aligned} q(x) &< \bar{q}(\bar{x}) < \bar{\bar{q}}(\bar{\bar{x}}) \\ q(x) &\leq 0 \\ \rho(x) &\geq 0 \\ \bar{\rho}(\bar{x}) &\leq 0 \end{aligned} \tag{3.26}$$

where q, $\bar{q}$, $\bar{\bar{q}}$ and $\bar{x}$ and $\bar{\bar{x}}$ are defined by equations (3.13, 14).

It is shown in [5] that the necessary conditions stated above are also sufficient for the existence of a solution of the type considered, provided the initial prescriptions are expandable in power series on each side of point 0. The lengthy and involved proof is omitted for reasons of brevity.

The conditions for the two previously excluded situations, i.e. $V(t) = -\beta$ and -1 are indicated in Table 1(b) in the strips at the top and bottom, respectively, of the portion of the table covered by Case 7.

A simplified set of sufficient conditions in terms of first derivatives may be obtained from equations (3.26), and is shown in Table 1(a). The velocity $V = V(t_0)$, in terms of first derivatives is the appropriate root of the quadratic equation

$$[\bar{m}' - \bar{\rho}' + (\rho - m)']V^2 + 2[\bar{m}' - \beta^2\bar{\rho}' + (\rho - m)']V + [\beta^2(\bar{m}' - \bar{\rho}') + (\rho - m)'] = 0. \quad (3.27)$$

In conclusion, it is noted that the discontinuities in the density and velocity at the interfaces vary as functions of the time t, a situation to be expected for a non-characteristic shock. Since the initial prescription at $t = t_0$ is continuous, the discontinuities begin with a zero value at point 0.

(c) *Review of the above results*

Based on an inverse approach, sets of conditions have been found which define continuous initial prescriptions for which solutions of the assumed nature exist, namely solutions with just one interface in a particular sector. It is helpful at this point to review the results in order to find the total range of initial prescriptions for which the above solutions apply.

Because of the simplicity of the alternate requirements in terms of first derivatives, the situation listed in the non-shaded portion of Table 1(a), is discussed first. (The shaded portion of the table is based on cases not yet considered.) The table, which is self-explanatory, shows the range of applicability of the various cases as functions of the parameters m'/ρ' and $\bar{m}'/\bar{\rho}'$. It is seen that for $m'/\rho' < 1$, Cases 3, 7 and 5 apply for the entire range of values of $\bar{m}'/\bar{\rho}'$ without overlap. However, the table does not disclose the situation on the boundaries between the cases. It is further seen from Table 1(a) that Case 1 applies for a portion of the range $m'/\rho' > 1$, while none of the Cases 1–7 apply for initial prescriptions for which $m'/\rho' > 1$ and $\bar{m}'/\bar{\rho}' < -\beta$, i.e. in the shaded portion.

To obtain a summary of the applicability of the various cases when the complete necessary and sufficient requirements are used, only initial prescriptions where $\rho(x) > 0$ and $\bar{\rho}(x) < 0$ are considered in Table 1(b). (The details when $\rho(x) = 0$ and/or $\bar{\rho}(x) = 0$ are listed in [5].) Table 1b shows that there is just one applicable case for each combination of the functions q, $\bar{q}$ and $\bar{\bar{q}}$ for which a solution has been shown to exist. Further, a solution is always found except when, simultaneously, $q(x) > 0$ and $\bar{q}(x) > 0$. This open region is shaded in Table 1(b), and corresponds to the shaded region in Table 1(a).

Table 1(b) clarifies the situation at the boundaries between the cases, when inequalities between two of the descriptive functions degenerate into equalities. For example, the boundary between Case 3 and Case 7 is given by $\bar{q}(\bar{x}) = q(x)$. If this relation holds, the table indicates that a solution according to Case 7 applies.

If the initial prescriptions satisfy any of the conditions obtained, the solution may be continued from $t = t_0$ to $t = t_0 + \Delta t$. The proof that the continuation exists is based on the premise of continuous and monotonic behavior, and for Case 7, on the possibility of expanding m, ρ, etc., near point 0. The requirements are thus again applicable for the continuation of the solution beyond $t = t_0 + \Delta t$ provided the premises are satisfied. In Cases 1, 3, 5 and 6 it may be shown in a nearly trivial manner that the solution at $t = t_0 + \Delta t$ satisfies the premises. In Case 7 the solution becomes discontinuous, a situation yet to be considered.

Finally, it is noted that there is a range of initial prescriptions corresponding to the shaded regions in Tables 1(a) and 1(b) for which the premise of one interface does not lead to a solution. Further, for uniqueness, it remains to be shown that no alternate solutions exist if the requirements listed in Table 1(b) hold.

4. REGIONS WITH SEVERAL INTERFACES, DISCONTINUOUS INITIAL PRESCRIPTIONS, CLOSURE AND USE OF THE APPROXIMATION $\beta = 0$

The first step in the treatment of cases with multiple interfaces is the recognition of the fact that no more than one interface may occur in each of the Sectors I through V defined by Fig. 3. This is proved in [5] by showing that the presence of two interfaces in any sector leads to contradictions. An upper limit is thus obtained on the number of cases which have to be investigated. The investigation [5] shows that, allowing for symmetry, actually only nine cases with two interfaces and four with three interfaces are possible.

Regions containing three interfaces

It was found in the preceding section that there are continuous initial prescriptions with $\bar{\rho}(\bar{x}) < 0$, $\rho(x) > 0$ where no solution with one interface exists. It is demonstrated in [5] that solutions with three interfaces are required for such prescriptions, and that these solutions do not furnish alternatives to the cases studied in Section 3. The possible cases with three interfaces are shown in Fig. 7 and the conditions for their occurrence are included in Table 1(a).

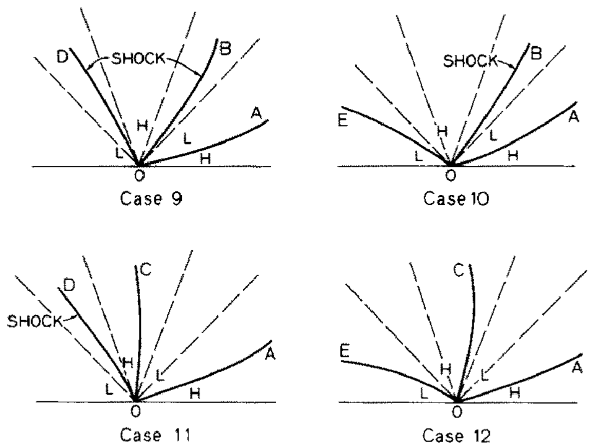

FIG. 7.

Regions containing two interfaces

If the initial prescriptions on the density to the right (R) and to the left (L) of the point 0 considered are similar, i.e. either $\rho_R > 0$ and $\rho_L > 0$, or $\rho_R < 0$ and $\rho_L < 0$, while $\rho(x_0) = 0$, the continuation may contain in general either no interface, or two interfaces [5]. The case $\rho_R > 0$, $\rho_L > 0$ is of considerable interest because it describes the initiation of a cavitated region. It was found in [5] that four basic possibilities having two interfaces may occur, shown in Fig. 8, and the conditions for each were determined. These conditions are listed in Table 2.

For initial prescriptions $\bar{\rho}_R < 0$, $\bar{\rho}_L < 0$ it is shown in [5] that five possibilities having two interfaces exist. These are not given in detail because this case concerns the rather rare formation of an uncavitated region within a cavitated one.

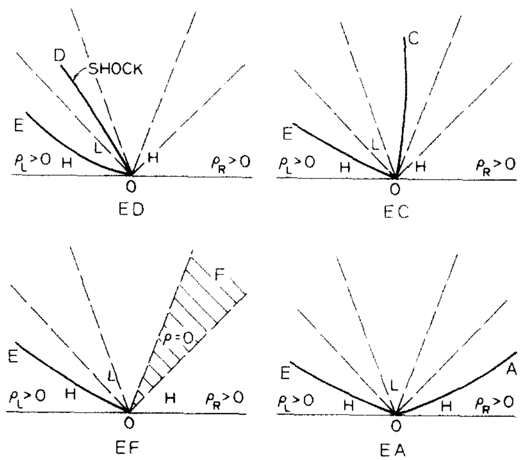

FIG. 8. (Letters ED, EC, etc., denote the cases listed in Table 2.)

TABLE 2. NECESSARY AND SUFFICIENT CONDITIONS PROVIDED $\rho_R(x_R) > 0, \rho_L(x_L) > 0$

$q_R(x_R) < 0$	$q_R(x_R) = 0$	$q_R(x_R) > 0$	
EC: $q_R(x_R) > \bar{q}_T(\eta)$; ED: $q_R(x_R) \le \bar{q}_T(\eta)$	EF	AE	$q_L(x_L) < 0$
No Interface		AG	$q_L(x_L) = 0$
		AC / AB (see below)	$q_L(x_L) > 0$

The cases listed in this table are illustrated in Fig. 8. The quantities q_R, q_L, η, and $\bar{q}_T$ are

$$q_R(x_R) = m_R(x_R) - m_0 - \rho_R(x_R), \quad x_0 < x_R \le x_0 + \epsilon_R$$

$$q_L(x_L) = m_L(x_L) - m_0 - \rho_L(x_L), \quad x_0 - \epsilon_L \le x_L < x_0$$

$$\eta = \frac{2\sqrt{2}\,\beta}{1 - \beta^2}(x_R - x_0)$$

$$\bar{q}_T(\eta) = \bar{m}_T(\eta) - m_0 + \beta\bar{\rho}_T(\eta)$$

The boundary between Cases AC and AB may be obtained by consideration of symmetry from the boundary between Cases EC and ED.

Continuation of the solution along an interface with discontinuity

It has previously been found that discontinuous conditions may occur, Case 7. At such an interface, Fig. 5, the relations (3.15, 16) apply. The premise that these relations hold for $t \leq t_0$ insures that the discontinuous prescriptions for $t = t_0$ satisfy the shock relation

$$(m-\bar{m})^2 = (p-\bar{p})(\rho-\bar{\rho}). \tag{4.1}$$

To avoid the discussion of special cases which occur when ρ or $\bar{\rho}$ vanishes, only initial prescriptions $\rho > 0 > \bar{\rho}$ are considered. Equation (3.16) then gives a value for the velocity $V \equiv V(t_0)$ in the range $\beta < V < 1$, thus permitting continuation of the solution by means of the characteristic relations (3.18) in combination with the shock relations (3.19). The latter are identical with equations (3.15, 16) except for the subscripts.

Having shown that the inequality $\beta < V < 1$ holds, the proof of existence given in [5] for Case 7 applies again. While the proof of uniqueness for Case 7 remains applicable and thus indicates that only one solution of the type considered here exists, uniqueness for the present case has not been fully proved in [5]. Completion of the proof would require demonstration that other configurations do not furnish alternative solutions.

Closure

The cases of multiple interfaces discussed above apply to situations in which a low-density region forms within a high-density region, or vice-versa. The alternative situation in which an existing region disappears because two interfaces converge has also been considered in [5]. It was found that in all cases at least one of the interfaces can be constructed, through use of the methods previously employed, prior to the determination of the point of closure and independently of the second interface. This second interface may then be constructed by employing the initial prescription together with the data on the first interface. Therefore no difficulty arises in the determination of the point of closure.

Boundary conditions

In physical problems, boundary conditions may be prescribed in conjunction with initial prescriptions, and it is necessary to consider the construction of the solution near a boundary. If no interface occurs at the boundary, the construction of the solution presents no difficulty for $\beta > 0$ because the usual characteristic construction near boundaries may be used. In addition, intersections of interfaces with boundaries may occur and can be treated for $\beta > 0$ by the methods developed in [5]. If $\beta = 0$, however, the situation near a boundary requires further consideration, see [5].

Use of the approximation $\beta = 0$ when $\beta \ll 1$

The justification for the use of a bilinear model to represent a fluid in cavitation problems is based on the fact that the cavitation pressure is expected to vary much less (as a function of the density) than the pressure in the uncavitated region. This implies that β is very much smaller than unity, and suggests that a numerical analysis might be based on the limiting value $\beta = 0$, leading to a somewhat simpler procedure. The applicability of the approximation $\beta = 0$ is investigated (and confirmed) in [5] under the following restrictions:

(1) Cavitated (low-density) regions which occur are bounded only by interfaces, i.e. cavitated regions do not come into contact with a physical boundary, such as the surface of a solid.

(2) The refined analysis based on $\beta \neq 0$ does not lead to an interface in the central sector, Sector III in Fig. 3.

When the above two conditions are satisfied, the use of the value $\beta = 0$ leads to no difficulties, and the results can be expected to be an approximation, the accuracy of which is a function of the magnitude of β.

5. NUMERICAL EXAMPLE, COMMENTS ON REFERENCES [3] AND [4], AND CONCLUSION

(a) *Numerical example*

The results obtained in the preceding sections were used to determine the response of a horizontal layer of mass on the surface of a half-space of fluid, Fig. 9. A plane pressure

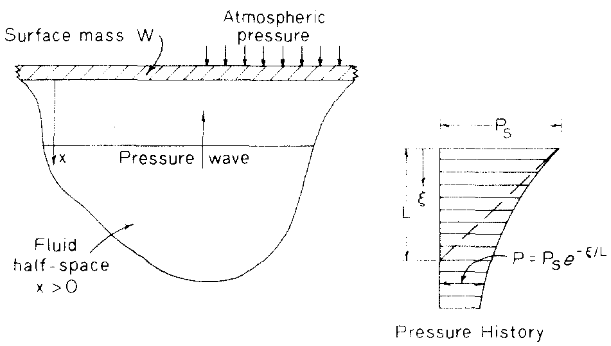

FIG. 9.

wave with a sudden rise and an exponential decay moves toward the surface, reaching the mass at the time $t = 0$. The system is subject to gravity and atmospheric pressure, all particles being at rest prior to arrival of the shock. The analysis is based on the degenerate model with $\beta = 0$, the applicability of which will be verificd subsequently, using the criteria given in Section 4.

The parameters used in the problem are

sound speed in the liquid, $c_1 = 4670$ ft/sec
sound speed in the cavitated region, $c_2 \approx 0$
peak value of the pressure wave, $P_s = 103$ psia
decay length of the pressure wave, $L = 4.74$ ft
atmospheric pressure, $P_A = 14.7$ psia
mass density of liquid, $\gamma_0 = 1.94$ slugs/ft^3
gravity, $g = 32.2$ ft/sec^2
surface mass per unit of area, $W = 0.921$ slugs/ft^2
cavitation pressure, $P_0 = 0$.

The differential equations are equations (2.5, 6, 7) with $\beta = 0$, while the initial and boundary conditions are, respectively,

$$\rho(x, 0) = p(x, 0) = \frac{P_S}{\gamma_0 c_1^2} e^{-x} + \frac{gL}{c_1^2} x + \frac{P_A}{\gamma_0 c_1^2} + \frac{Wg}{\gamma_0 c_1^2}$$

$$m(x, 0) = -\frac{P_S}{\gamma_0 c_1^2} e^{-x} \tag{5.1}$$

and

$$\dot{m}(0, t) = \frac{\gamma_0 L}{W}\left[\frac{P_A}{\gamma_0 c_1^2} - p(0, t)\right] \tag{5.2}$$

where $x = X/L$ and $t = c_1 T/L$.

The numerical solution was obtained by a characteristic approach. The history of the cavitated region is shown in Fig. 10. No cavitation occurs until $t = t_X$, at which time the requirements for the combination AE of Fig. 8 apply, i.e. the boundary of the cavitated region during opening consists of two branches, XY and XZ, each being of the type treated as Case 1. The simple form of the incoming wave permits the determination of a closed solution $f(x, t) = 0$ for the portion YXZ of the boundary. At the terminal points Y, Z the opening interfaces become tangent to the characteristics ± 1.

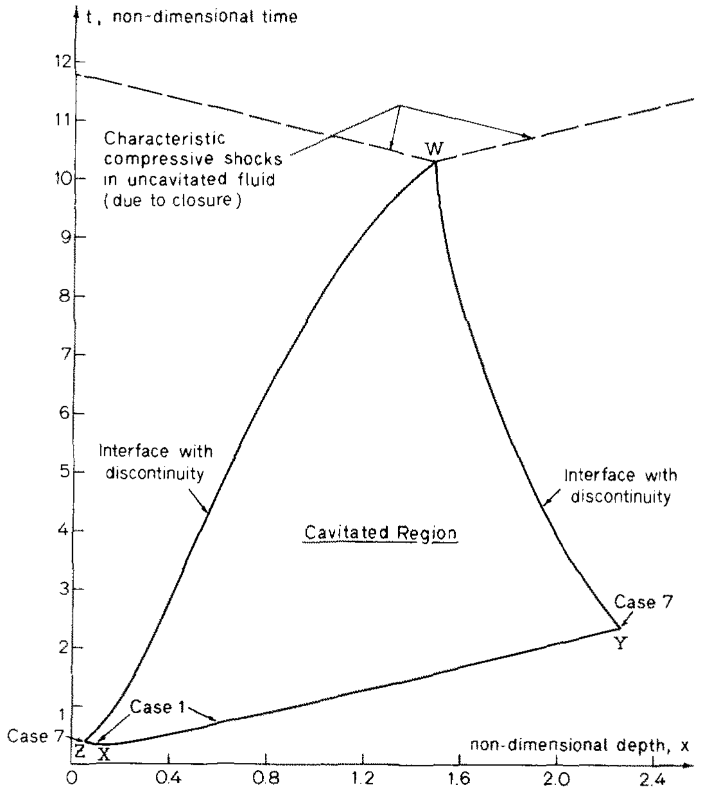

FIG. 10. Time-history of cavitated region.

The continuation of the solution near point Y depends on the values of m and ρ for $t = t_Y$. These values are such that a discontinuous interface, Case 7, develops. The change in the type of the interface is due to the fact that the value of the derivative of $(m - \rho)$ to the right of the interface changes sign at $t = t_Y$. For $t < t_Y$ the sign is positive, as appropriate for Case 1, while for $t > t_Y$ the sign is negative, as appropriate for Case 7. At $t = t_Y$ the character of the solution is governed by higher (second) derivatives, which indicate the applicability of Case 7, the velocity of the interface being $|V_Y| = \frac{1}{5}$. A similar situation applies at point Z.

At the time $t = t_W$ the two interfaces intersect, leading to closure. The impact of the two bodies of liquid at the time $t = t_W$ generates two compressive shock waves in the (subsequently uncavitated) fluid for $t > t_W$. One of the shocks at a later time reaches the surface, so that a reflected wave of underpressure ensues. However, the intensities are low, and no further cavitation occurs for $t > t_W$.

Before discussing the behavior of the mass, the applicability of the approximation $\beta = 0$ is verified. The cavitated region did not reach the boundary $x = 0$, so that the first restriction stated for the use of $\beta = 0$ is satisfied. Further, the interface configurations found by use of the value $\beta = 0$, Cases 1, 7 and the continuation of the interface with discontinuity, can be shown to be of the same types for other, small values of β. The second requirement for the use of $\beta = 0$ is thus also satisfied. It can therefore be asserted that the solution obtained is a good approximation to the solution which would be found by use of the theory based on equations (2.5, 6, 7) with $\beta \neq 0$, provided the actual value of β is small compared to unity.

It is also necessary to confirm that the linearization leading to equations (2.5, 6, 7) is permissible for the data used in the example. This requires $|\rho| \ll 1$ and that the magnitude of the particle velocity is everywhere small in comparison to the local sound speed, even in the cavitated region. The non-dimensional value, u, of the particle velocity in the example may be determined from the last of equations (2.6). Its magnitude,

$$|u| \approx \left| m + \frac{gL}{c^2} \right| \tag{5.3}$$

should be less than the actual value of β. The values of $|\rho|$ and $|u|$ were found not to exceed 2×10^{-3} and 10^{-3}, respectively. The condition on $|\rho|$ is thus clearly satisfied. In the absence of information on the actual value of β, one can only say that the linearization is appropriate if β is sufficiently larger than 10^{-3}.

For structural purposes the motion of the mass is of interest. The history of its velocity is shown in Fig. 11. This figure also indicates the history if cavitation in the fluid is ignored, i.e. if tension in the fluid is permitted. It is seen that the mass reaches its peak velocity prior to the onset of cavitation, so that this peak velocity is not affected by cavitation. However, if the displacement of the mass is considered, the solution allowing for cavitation will lead to a larger peak value of the displacement, because subsequent to the onset of cavitation the positive velocity is larger. The difference in the peak displacement in this example is substantial, about 40%.

An interesting phenomenon in the history of the velocity of the mass occurs at the time $t = t_s$ when the secondary shock, generated by the closure of the cavitated region at $t = t_W$, reaches the surface. This shock causes the very rapid drop in the absolute value of the velocity subsequent to $t = t_s$.

Characteristic methods, although quite appropriate in one-dimensional problems become unwieldy in multi-dimensional problems. In such situations finite difference schemes are appropriate. Such schemes can be tested against the above numerical result. A finite difference scheme based on the method of Lax [6] was applied to the above example. This scheme approximates equations (2.5, 6, 7, 5.1, 2) by means of central space differences and forward time differences in a staggered grid. The numerical solution obtained gave

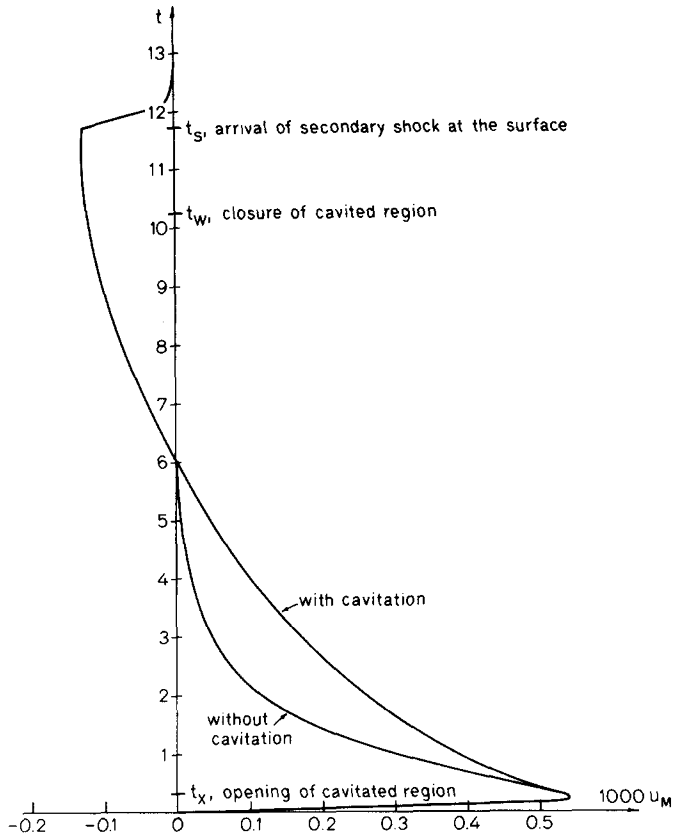

FIG. 11. Non-dimensional upward velocity of mass, U_M.

excellent results for grid ratios $\Delta t/\Delta x \leq 1$ combined with grid sizes smaller than $\Delta x = \Delta X/L = 0{\cdot}01$. An extension of this scheme appears thus suitable for multi-dimensional problems.

(b) *Comments on references* [3] *and* [4]

Reference [3] treats a problem which is quite similar to the one treated in (a), except that the layer of mass is elasto-plastically supported. Making the assumption that the cavitation pressure is zero, the opening of the cavitated region obtained in the reference is in complete agreement with the present analysis for $\beta = 0$. In the subsequent closing stage, Ref. [3] disregards all compressibility and wave propagation effects, so that only a crude approximation is obtained.

Reference [4] is not concerned with structural response. It contains the reflection of pulses from a surface solely under the action of gravity and atmospheric pressure. The shape of the pulses is similar to the one used in the above example. This reference also assumes that the cavitation pressure vanishes, but assumes a tensile breaking stress in the liquid which must be exceeded prior to cavitation. For the situation considered, Ref. [4] shows as a first step that this assumption does not lead to cavitation in the usual sense, but to layers of liquid separated by voids. The reference proceeds to approximate these alternate layers by a cavitated region, by using a limiting process based on the assumption that the breaking strength is low compared to the stress level. For the problem considered this process leads to opening velocities of the cavitated region which agree with the ones found from the present analysis. In the later stages, when the cavitated region closes, Ref. [4] utilizes the approximate approach of Ref. [3]. A comparison indicates that the cavitated region closes according to Ref. [4] much later than according to an analysis based on the present approach. It is important to note that this difference is not due to the introduction of the breaking strength, but solely due to the omitted effects of wave propagation during the collapse of the cavitated region.

In Ref. [8], a continuation of the work in Ref. [4], numerical results on the opening in three-dimensional situations are reported. In addition, planned work on the closing stage is outlined, which includes allowance for the shocks generated at closing due to compressibility effects. As mentioned in the example, such shocks are obtained in the present analysis.

(c) *Conclusion*

The problem of one-dimensional wave propagation in a bilinear fluid has been considered by a characteristic approach, the emphasis being put on the determination of the interfaces. Excluding singular situations and discontinuous initial prescriptions which were not fully considered, it was found in [5] that solutions to the bilinear problem for $\beta > 0$ exist and are unique. It was further shown in [5] that a degenerate bilinear fluid, $\beta = 0$, Fig. 1(b), can be used with restrictions.

The results obtained for the one-dimensional case are also applicable to two dimensional steady-state problems, [9]. In such cases a transformation $x - Vt = \xi$ reduces the problem to one in two variables. This may be utilized to check two-dimensional computations based on finite difference methods.

The complications in treating two- or three-dimensional problems in a bilinear fluid by characteristic methods are so severe that finite difference approaches appear preferable. The fact that in the one-dimensional case for $\beta \neq 0$ no difficulty with uniqueness and existence was found, leads to the expectation that a numerical approach on the same basis will be permissible. However, as indicated earlier, the use of the simplification $\beta = 0$ may not be permissible without restrictions and requires further investigation.

REFERENCES

[1] H. H. BLEICH, Dynamic interaction between structures and fluid. *Structural Mechanics*, p. 263. Pergamon Press (1960).

[2] E. H. KENNARD, Cavitation in an elastic liquid. *Phys. Rev.* **63**, 172 (1943).

[3] H. N. V. TEMPERLY, Theoretical investigation of cavitation phenomena occurring when an underwater pressure pulse is incident on a yielding surface—I. *Underwat. Expl. Res.* **3**, 255 (1950).

[4] V. CUSHING, Study of bulk cavitation and consequent water hammer. Final Report, Engineering Physics Company, Rockville, Maryland. Office of Naval Research, U.S. Navy, Contract Nonr-3389(00), (1961).

[5] H. H. BLEICH and I. S. SANDLER, Dynamic interaction between structures and bilinear fluids, Columbia University, Office of Naval Research Contract Nonr 266(86), Technical Report No. 43, April 1968.

[6] P. D. LAX, Weak solutions of nonlinear hyberbolic equations and their numerical computation. *Communs pure appl. Math.* **7**, 159–193 (1954).

[7] L. D. LANDAU and E. M. LIFSHITZ, *Fluid Mechanics*. Addison-Wesley (1959).

[8] V. CUSHING, G. BOWDEN, and D. REILY, Three-Dimensional Analysis of Bulk Cavitation-Interim Report, Engineering Physics Company, Rockville, Maryland. Office of Naval Research, U.S. Navy, Contract Nonr 3709(00), (1962).

[9] H. N. V. Temperley, Theoretical investigation of cavitation phenomena occurring when an underwater pressure pulse is incident on a yielding surface—III. *Underwat. Expl. Res.* **3**, 283 (1950).

(*Received* 20 *February* 1969; *revised* 7 *August* 1969)

Абстракт—В смысле математической модели жидкостей, у которых могут появлятся кивитеции исследуется распределение волны в билинейных жидкостях. Основной трудностью является определение априори неизвестных и движущихся пределов между двумя стадиами жидкости.

Используется характеристический подход для исследования одномерных задач. Даже для так относительно элементарных случаев, может появлятся значительное их количество; необходимый также громоздкий анализ. Работа обсуждает также нетривиальные вопросы единственности и существования решений для билинейной модели.

Представленный процесс применяется для типичного примера.